On the relationship between the angular deficit and the internal structure of straight cosmic strings.

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We give a brief discussion on the limitations involving the expression $\mu = \Delta\phi/8\pi$ ($G = c = 1$), which relates the string linear energy density μ to the conical deficit angle $\Delta\phi$. Then, we establish a new equation between the angular deficit and a combination of the components of the string stress-energy tensor which shows that the angular deficit is determined not only by the amount of proper matter of the string but also, in a Newtonian sense, by its internal gravitational field.

Cosmic strings are topological defects [1] which produce very interesting gravitational effects. We find in the literature a great number of articles dealing with this subject (see Ref. [1] for a extensive reference list). In this paper we want to focus our attention on a specific aspect of cosmic strings. It is well known that the asymptotic space-time generated by an infinite straight cosmic string has no curvature, but has an angular deficit [1,2]. It is argued in many works that this angular deficit can be connected with a physical property of the string, namely, its "linear energy density" by the expression $\mu = \triangle\phi/8\pi$ [1–4]. Thus, it would be possible to infer the cosmic string linear energy density measuring the space-time angular deficit. We would like to investigate this question more closely.

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The cosmic string in a curved space-time is a special solution of the Einstein equations coupled with scalar and gauge fields in which the stress-energy tensor is concentrated around a "line" of false vacuum. The non-gravitational fields are described by the Lagrangian [2]:

$$\mathcal{L} = -\frac{1}{2}\nabla^{\nu}R\nabla_{\nu}R - \frac{1}{2}R^2(\nabla_{\nu}\psi + eA_{\nu})(\nabla^{\nu}\psi + eA^{\nu}) \\ -\alpha\left(R^2 - \eta^2\right)^2 - \frac{1}{16\pi}F_{\nu\kappa}F^{\nu\kappa}, \tag{1}$$

where A_ν is a vector field, $\Phi \equiv R\,e^{i\psi}$ is a complex scalar field, ∇_ν is the covariant derivative with respect to the space-time metric, $F_{\nu\kappa} \equiv \nabla_\nu A_\kappa - \nabla_\kappa A_\nu$, α, η and e are constants. (Note that $\nu, \kappa = 0, 1, 2, 3$, and $\hbar = 1$). Actually, there are alternatives proposals for string models [1], but here we are considering the Abelian-Higgs model (1), which is the simplest one.

The exact string solution for the Lagrangian (1) is not known but it is assumed that the fields have the form:

$$R = R(r), \tag{2}$$

$$\psi = \phi, \tag{3}$$

$$A_\nu = \frac{1}{e}\left[P(r) - 1\right]\nabla_\nu\phi. \tag{4}$$

It is expected that $R \to \eta$ and $P \to 1$ as $r \to \infty$. This asymptotic behavior of the fields ensures that the stress-energy tensor drops to zero far from the string axis [2]. Other relevant property, which comes from (2),(3) and (4) is that the associated stress-energy tensor is static, has cylindrical symmetry and it is boost-invariant along the axis direction. It is reasonable to assume the same symmetries for the metric. The most general metric with these symmetries can be put in the form:

$$ds^2 = e^{2V(r)}(-dt^2 + dz^2) + dr^2 + U^2(r)d\phi^2, \tag{5}$$

where $-\infty < t, z < \infty$, $0 < r < \infty$ and $0 < \phi < 2\pi$. The regularity condition on the string axis imposes that $\frac{U}{r} \to 1$ as $r \to 0$ and that $V(r)$ must be a well behaved function on the axis.

According to Garfinkel [2] and other authors [1,5,6], the linear energy density μ of the string is defined as the integral of the energy density $\rho = -T^t_t$, which is measured by the observers associated with the vector field $e^{-V(r)}\partial_t$, over the transversal section $t = const$ and $z = const$:

$$\mu \equiv -\int\int T^t_t U(r) dr d\phi. \tag{6}$$

With some additional assumptions, Garfinkel [2] has shown that the metric (5) generated by the cosmic string is asymptotically a conical space-time with $UV' \to 0$ as $r \to \infty$. Taking into account these results and using the Einstein equation, $8\pi U T^t_t = U'' + (UV')' + UV'^2$, one can establish, from definition (6), a relation between μ and the angular deficit $\triangle\phi \equiv 2\pi[1 - U'(\infty)]$ of the conical space:

$$\mu = \frac{\triangle\phi}{8\pi} - \frac{1}{4}\int UV'^2 dr. \tag{7}$$

When the energy of the false vacuum is null ($\eta = 0$), the metric reduces to the Minkowski metric. Then, Garfinkel [2] argued that for small η^2 (the estimated value for grand unified strings is $\eta^2 < 10^{-4}$), the metrics components will have little deviation from the flat metric values. Based on this approximation, he estimated the order of the correction for U and V and, then, showed that μ is of the order of η^2 but the integral term is of the order of η^4. Thus, to a good approximation:

$$\triangle\,\phi \simeq 8\pi\mu. \tag{8}$$

The complicated form of the full Einstein-gauge-scalar equations for the strings does not recommend that we look for the exact string solution. Thus, we have to restrict ourselves to approximated relations such as equation (8). To get further insight about the exact form of the metric in the string solutions, some authors propose to use simple models to describe cosmic strings.

It is estimated that the transverse dimensions of the string is very small compared with its length. Then, it may be appropriate treating the string as an energy distribution concentrated around an axis in such a way that the string can be confined in a thin cylinder. In

addition, it would be reasonable to assume that the stress-energy tensor has all the symmetries mentioned previously. Hiscock [3] and Gott [4] studied the simplest model with these properties. In their model the stress-energy tensor has an uniform energy density ρ and it is given by

$$T_t^t = T_z^z = -\frac{\gamma^2}{8\pi\ell^2} = const, \qquad (r \leq \ell), \tag{9}$$

with all other components equal to zero. The parameter ℓ is the cylinder radius. For $r > \ell$, the stress-energy vanishes, since we are considering that the string is surrounded by vacuum.

The space-time generated by this energy distribution can be determined. We can distinguish between two regions, inside and outside the cylinder, to which correspond respectively an interior and an exterior metric that can be smoothly joined :

$$ds^2 = -dt^2 + dr'^2 + (\ell/\gamma)^2 \sin^2(\gamma r'/\ell)\, d\phi^2 + dz^2, \qquad r' \leq \ell \tag{10}$$

$$ds^2 = -dt^2 + dr^2 + \lambda^2 r^2 d\phi^2 + dz^2, \qquad r > r(\ell), \tag{11}$$

with $\lambda = \cos\gamma$ and $r(\ell) = (\ell/\gamma)\tan\gamma$. These latter relations come from the continuous junction condition through the cylinder surface $r' = \ell$ (or $r = r(\ell)$) , which demands the metric and extrinsic curvature of the cylinder surface to be both continuous.

Regarding the definition of the linear energy density, it can be shown that the angular deficit is

$$\Delta\phi = 8\pi\mu, \tag{12}$$

which coincides with the approximate result (8), but now holds for all orders in μ. Note that this equation is independent of the cylinder radius. Then, taking the limit $\ell \to 0$, the distribution can be idealized as a line source with a well defined linear energy density which produces a conical space-time with an angular deficit proportional to μ.

The generality of this result seems to be questionable, since it was derived from a string whose stress-energy tensor has a very simple form. But the strings models can be enlarged as was demonstrated by Linet [5], who showed that for models with non-uniform energy

the relation (12) is also satisfied. Then, we could be led to suppose that this is a general characteristic, i.e., that the angular deficit of a conical space-time generated by a line source is given by (12).

However this conclusion is not correct as was shown by Geroch and Traschen [7], who considered another string model for which the interior metric corresponds to (10) multiplied by a conformal factor:

$$ds^2 = \exp[2\beta f(r'/\ell)]\,[-dt^2 + dr'^2 + (\ell/\gamma)^2 \sin^2(\gamma r'/\ell) d\phi^2 + dz^2], \qquad (r' \leq \ell) \tag{13}$$

where f is a smooth non-negative function with compact support inside the interval $\left[\frac{1}{2}, 1\right]$. The properties of the function f at $r' = \ell$ ensure a continuous junction with the exterior conical metric. Moreover, the associated stress-energy tensor has all desirable symmetries. Hence, the metric above is, in fact, a possible interior solution. Now, calculating the linear energy density we obtain

$$\mu = \frac{\triangle\phi}{8\pi} - \frac{\beta^2}{4\gamma} \int_{\frac{1}{2}}^{1} \sin(\gamma x)\, [f'(x)]^2\, dx. \tag{14}$$

We can choose β in such a way that μ is positive. Then, taking the limit $\ell \to 0$, we get a line source for the conical metric with μ less than the angular deficit. Thus, according to Geroch and Traschen [7]: "the procedure of taking the limit of a family of well-behaved sources does not in general yield an unique relationship between the mass per unit length and deficit angle". This result was investigated and confirmed by other authors who have shown that, except in some special case when the constants of the Lagrangian e and α satisfy a particular tuning [8], there is no simple relation between μ and $\Delta\phi$ in the thin limit of cosmic strings [9,10].

So, we have to specify stronger conditions for the string models, imposing a great number of constraints to the internal distribution, if we want to ensure the validity of relation (12), see Refs. [8,9,11]. Another possibility is to restrict the space-time to a special class of manifolds for which the distributional formalism can be used to treat conical singularities [12].

We would like to approach this problem in another way. Instead of trying to determine the conditions to be satisfied by the internal model in order to preserve relation (12), we admit that (12) may be incomplete and ask for the existence of another physical attribute of the source that could be unambiguously associated with the angular deficit.

Firstly, consider the metric in the Gaussian form:

$$ds^2 = dr^2 + g_{ij}dx^i dx^j, \qquad i,j = 1,2,3 \tag{15}$$

The Einstein equations in this coordinate system can be written in the following way:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial r}\left[\sqrt{-g}\left(K^i_j - \delta^i_j K\right)\right] + \left({}^{(3)}R^i_j - \delta^i_j\,{}^{(3)}R\right) = -8\pi\left(T^i_j + \delta^i_j T^r_r\right), \tag{16}$$

$$K_{,i} - {}^{(3)}\nabla_j K^j_i = -8\pi T^r_i, \tag{17}$$

$$K^i_j K^j_i - K^2 - {}^{(3)}R = -16\pi T^r_r, \tag{18}$$

where $K^i_j = \frac{1}{2}g^{ik}g_{kj,r}$ is the extrinsic curvature of the $r = const$ hypersurfaces, K is the trace of K^i_j, and ${}^{(3)}R^i_j$ is the Ricci tensor of the hypersurface $r = const$.

We consider an internal model whose associated stress-energy tensor has all characteristics mentioned previously. So, we will admit that the interior metric can be put in the form (5) with the same regularity conditions. The extrinsic curvature of the $r = const$ hypersurface is easily calculated: $K^t_t = K^z_z = V'$ and $K^\phi_\phi = U'/U$, where prime denotes derivative with respect to r. For this metric, the Einstein equations reduce to the form:

$$\frac{\partial}{\partial r}\left[\sqrt{-g}\left(K^i_j - \delta^i_j K\right)\right] = -8\pi\sqrt{-g}\left(T^i_j + \delta^i_j T^r_r\right), \tag{19}$$

$$0 = T^r_i, \tag{20}$$

$$K^i_j K^j_i - K^2 = -16\pi T^r_r, \tag{21}$$

Integrating equation (19) over the transversal section of the string, the surface $t = const$ and $z = const$, we obtain:

$$\sqrt{-g}\left(K^i_j - \delta^i_j K\right)\Big|_0^\ell = -4\int_0^{2\pi}\int_0^\ell \sqrt{-g}\left(T^i_j + \delta^i_j T^r_r\right) dr d\phi. \tag{22}$$

We are looking for interior metrics that can be matched smoothly with a conical metric. On the cylindrical hypersurface $r=\ell$, the continuity conditions for the metric and extrinsic curvature requires that:

$$V(\ell)=0; \tag{23}$$

$$V'(\ell)=0; \quad U'(\ell)=\lambda. \tag{24}$$

Then, from these conditions and equation (22), we obtain

$$\left(\frac{\Delta\phi}{8\pi}-\chi\right)\left(\delta^i_{\ j}-\delta^i_{\ \phi}\,\delta^\phi_{\ j}\right)=-\int_0^{2\pi}\int_0^{\ell}\sqrt{-g}\left(T^i_j+\delta^i_jT^r_r\right)drd\phi \tag{25}$$

where $\chi=\left(1-e^{2V(0)}\right)/4$.

Now in order to write χ in terms of the physical attributes of the cosmic string let us consider constraint equation (21) and the equation of energy-momentum conservation $\nabla_\kappa T^\kappa_{\ \nu}=0$:

$$V'^{\,2}+2V'\frac{U'}{U}=8\pi p_r \tag{26}$$

$$p_r'+2\left(p_r+\rho\right)V'+\left(p_r-p_\varphi\right)\frac{U'}{U}=0 \tag{27}$$

From these equation we can show that

$$V\left(0\right)=\int_0^{\ell}\left[\frac{p_r'-\sqrt{p_r'^{\,2}-8\pi p_r\left(p_r-p_\varphi\right)\left(3p_r+4\rho+p_\varphi\right)}}{\left(3p_r+4\rho+p_\varphi\right)}\right]dr \tag{28}$$

where $\rho=-T^t_t$, $p_r=T^r_r$ and $p_\phi=T^\phi_\phi$ are the volumetric energy density, the radial pressure and the hoop pressure respectively, measured by observers associated to the field $e^{-V}\partial_t$.

For $i,j=t$, we have

$$\frac{\Delta\phi}{8\pi}=\int_0^{2\pi}\int_0^{\ell}\sqrt{-g}\left(\rho-p_r\right)drd\phi+\chi. \tag{29}$$

For $i,j=\phi$, we have the well known constraint for cosmic strings [10]:

$$\int_0^{2\pi}\int_0^{\ell}\sqrt{-g}\left(p_r+p_\phi\right)drd\phi=0. \tag{30}$$

To avoid misunderstanding it is worthy to emphasize here that the above equation does not imply that the integral of each component T_r^r and T_ϕ^ϕ, taken separately, also vanishes. Thus, the additional terms that appear in (29) cannot be neglected. However, the integrals of the "Cartesian" components T_x^x and T_y^y (where $x = r\cos\phi$ and $y = r\sin\phi$) are both null [13,14]. At first sight this could seem contradictory, since the new terms in (29) depend on the transversal components of pressure whose integral, as we said before, vanish in Cartesian coordinates. This point can be clarified by noting that the expression (29) is not covariant, since it holds only in coordinates in which the Einstein equations assume the form (19), which can be directly integrated. In Cartesian coordinates the correspondent equations are not so simple, except in the linearized approximation. In this regime, as was shown in [14], for Abelian-Higgs cosmic strings, the deficit angular depends only on the linear energy density according to (6). But this result is exactly what we obtain from (29) in the linear approximation. Indeed, we can verify that, in this limit, $\chi \simeq \int_0^{2\pi}\int_0^{\ell}\sqrt{-g}p_r dr d\phi$, and hence the formula (29) reproduces equation (6).

Equation (29) is an exact formula which generalizes (12). We can notice important differences by comparing them. In (29) the integration is done taking the volume $\sqrt{-g}$ of the whole space-time, while definition (6) uses the area element $\sqrt{^{(2)}g}$ induced on the surface $t = const$ and $z = const$. This replacement has a physical meaning which we shall try to interpret now. The factor $\sqrt{^{(2)}g}$ gives only the correction for the area element $da = \sqrt{^{(2)}g}drd\phi$ due to the curvature of the surface $t = const$ and $z = const$. On the other hand, $\sqrt{-g} = \sqrt{-g_{tt}}\sqrt{^{(3)}g}$ contains two distinct contributions: one due to the spacial curvature incorporated by $dV = \sqrt{^{(3)}g}drd\phi dz$, which is the proper volume element of the three-space sliced by the observers $e^{-V}\partial_t$; and another contribution corresponding to the term $\sqrt{-g_{tt}}$. As we know, in the weak field approximation, this component g_{tt} is related to the Newtonian gravitational potential ψ ($g_{tt} \simeq 1 + 2\psi$). This is an interesting result because, expanding (29) in this approximation, we obtain a term proportional to $\frac{1}{2}\rho\psi$, which is the gravitational energy of the Newtonian field ψ generated by ordinary (non-relativistic) matter distribution. Strictly speaking, in the string case, there is no Newtonian limit properly, since the energy

distribution is ultra-relativistic, once the pressure and the energy are of the same order [2]. Because of this and the fact that the state equation $p_z = -\rho$ holds for the cosmic string, the analogous of the Newtonian potential satisfies $\nabla^2\psi = 4\pi\left(p_r + p_\phi\right)$, instead of the common result $\nabla^2\psi = -4\pi\rho$. Hence, the term $\frac{1}{2}\rho\psi$ is not exactly the Newtonian gravitational energy density. However, it shows that the gravitational potential also contributes to the angular deficit, not only the amount of matter, as it would be the case if g_{tt} were not present in the formula (29). Thus, in this sense, through the factor $\sqrt{-g}$ in (29), the gravitational field contributes itself to the angular deficit.

Besides this, the presence of the additional term related to radial pressure highlights the influence of the internal gravitational field on the angular deficit, since the static configuration is reached only after the equilibrium between the radial pressure and the gravitational force is attained. Finally, we have the term χ which is associated to the gravitational red shift between the axis and the hypersurface at ℓ. Thus, equation (29), which reminds us of Tolman's integrals [15], suggests that the gravitational field (in a Newtonian sense, as explained above) is also relevant for the exact relationship between the angular deficit and the internal structure of the source, not only the amount of matter as indicated in the relation (6).

The left hand side of equation (29) does not depend on the cylinder radius ℓ. Thus, this relation is maintained even if we take the limit $\ell \to 0$, whenever the integrals on the right hand side converge in this limit. Relation (29) is also applicable to extended distributions, as the complete string model described by the Lagrangian (1). In this case, the junction condition must be considered asymptotically at $\ell = \infty$.

For simple models in which $p_r = 0$, relation (29) reproduces the old one (6). However, equation (29) holds for more models than (12) does, since, the radial and hoop pressures do not have to be zero in all interior points.

Acknowledgment. I would like to thank Dr. Carlos Romero for suggestions and for a careful reading of this paper. I am grateful to the referees for their comments.

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